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IMPACT OF A CIRCULAR DISK ONTO A LIQUID OF SHALLOW DEPTH

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M. I. CHEBAKOV
(Rostov-on-Don)

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We consider the problem of the impact of an absolutely rigid disk onto the surface of an ideal liquid of shallow depth. The solution is obtained by reducing the dual integral equations, which arise in the problem, to an infinite system of linear algebraic equations. Expressions are obtained for determining the impact pressures, the apparent additional mass, and the apparent additional moment of inertia. We derive a condition for a nonseparable impact. This problem was investigated in [2] for the case of a liquid of great depth.

1. Statement of the problem. We take the origin of Cartesian (x, y, z) and cylindrical (r, θ, z) coordinate systems on the free surface of the liquid at the center of the disk with the z-axis directed normal to the free surface and pointing downwards into the liquid.

In the case of a centered impact the potential of the velocities acquired by the liquid particles is given by

$$\varphi(r, z) = \int_0^\infty f(\alpha) \frac{\operatorname{ch} \alpha (h-z)}{\operatorname{ch} \alpha h} J_0(\alpha r) \alpha \, d\alpha \tag{1.1}$$

Here h is the liquid depth, $J_0(\alpha r)$ is the Bessel function of the first kind of order zero, and $f(\alpha)$ is obtained from the following dual integral equation:

$$\int_0^\infty f(\alpha) \alpha^2 K(\alpha) J_0(\alpha r) \alpha \, d\alpha = -U, \quad r \leqslant a \tag{1.2}$$

$$\int_0^\infty f(\alpha) J_0(\alpha r) \alpha \, d\alpha = 0, \quad r > a \qquad \left(K(\alpha) = \frac{\operatorname{th} \alpha h}{\alpha}\right)$$

where U is the disk velocity and a is the disk radius. To obtain the velocity potential in the case of off-center impact it is necessary to add the following function (see [2]) to the potential $\varphi(r, z)$:

$$\psi(r, z) = \frac{\partial}{\partial x} \Phi(r, z) = \frac{\partial}{\partial x} \int_0^\infty F(\alpha) J_0(\alpha r) \frac{\operatorname{ch} \alpha (h-z)}{\operatorname{ch} \alpha h} \alpha \, d\alpha \tag{1.3}$$

Here $F(\alpha)$ is a solution of a dual integral equation, which differs from Eq. (1.2) in that $-U$ is replaced by $^1/_2\, \omega r^2 + c$, where ω is the angular rate of rotation of the disk and c is an arbitrary constant. We assume that the x-axis passes through the point where the impact occurs.

2. Solution of dual integral equations. We consider the more general dual integral equation

$$\int_0^\infty Q(\alpha) K(\alpha) J_n(\alpha r) \alpha \, d\alpha = J_n(\varepsilon r), \quad r \leqslant a, \qquad \int_0^\infty Q(\alpha) J_n(\alpha r) \alpha \, d\alpha = 0, \quad r > a \tag{2.1}$$

$$K(\alpha) = A\frac{P_1(\alpha^2)}{P_2(\alpha^2)} = A\prod_{n=0}^{\infty}\left(1+\frac{\alpha^2}{\delta_n^2}\right)\left(1+\frac{\alpha^2}{\gamma_n^2}\right)^{-1} \qquad (2.2)$$

Here $J_n(x)$ is the Bessel function of the first kind of order n, $i\delta_n$ and $i\gamma_n$ is a countable set of zeros and poles of the function $K(\alpha)$ lying in the upper halfplane. We assume that there are no multiple zeros and poles and that $\delta_n \neq \gamma_m$ $(n, m = 1, 2, \ldots)$. We assume also that δ_n and γ_n increase monotonically in absolute value as n increases, assuring thereby the convergence of the infinite product (2.2); in addition, we assume that on an arbitrary regular system of contours C_n $(C_n \subset C_{n+1})$ the following estimate is valid for $n \to \infty$:

$$K(\alpha) = O(|\alpha|^p), \qquad p \leqslant 0 \qquad (2.3)$$

Using the relation (2.2) and

$$L_x J_n(x\alpha) = \alpha^2 J_n(x\alpha), \qquad L_x = \frac{n^2}{x^2} - \frac{1}{x}\frac{d}{dx} - \frac{d^2}{dx^2} \qquad (2.4)$$

the dual equation (2.1) can be reduced to the form [1]

$$AP_1(L_r)q(r) = P_2(L_r)J_n(\varepsilon r), \ r \leqslant a, \ q(r) = 0, \ r > a \qquad (2.5)$$

$$q(r) = \int_0^\infty Q(\alpha)J_n(\alpha r)\alpha\, d\alpha \qquad (2.6)$$

Here $P_1(L_r)$ and $P_2(L_r)$ are differential operators with respect to r of infinite order. The solution of the differential equation from (2.5) for $q(r)$ can be written in the form [1]

$$q(r) = K^{-1}(\varepsilon)J_n(\varepsilon r) + \sum_{k=1}^{\infty}[C_k J_n(i\delta_k r) + D_k N_n(i\delta_k r)], \qquad r \leqslant a \qquad (2.7)$$

($N_n(x)$ is the Bessel function of the second kind; C_k and D_k are constants). Considering $q(r)$ to be bounded for $r \to 0$, we set $D_k = 0$ $(k = 1, 2 \ldots)$. Taking into account the inverse Hankel transformation and the second relation in (2.5), we have

$$Q(\alpha) = K^{-1}(\varepsilon)\int_0^a J_n(\varepsilon r)J_n(\alpha r)r\,dr + \sum_{k=1}^{\infty} C_k\int_0^a J_n(i\delta_k r)J_n(\alpha r)r\,dr \qquad (2.8)$$

We determine the constants C_k by having the solution (2.8) satisfy the dual equation (2.1). We can represent the meromorphic function $K(\alpha)$, subject to the assumptions we have made for it, in the form of a sum of principal values:

$$K(\alpha) = \sum_{m=1}^{\infty}\frac{b_m}{\alpha^2+\gamma_m^2}, \qquad b_m = 2i\gamma_m\{[K^{-1}(i\gamma_m)]'\}^{-1} \qquad (2.9)$$

We now substitute the relations (2.8) and (2.9) into the first of the relations (2.1). Taking into account the relations

$$\int_0^\infty \frac{1}{u^2+\gamma_m^2}J_n(ux)J_n(uy)u\,du = \begin{cases} I_n(x\gamma_m)K_n(y\gamma_m), & x<y \\ I_n(y\gamma_m)K_n(x\gamma_m), & y<x \end{cases}$$

$$\int_0^a I_n(\delta_k y)K_n(\gamma_m y)y\,dy = a\frac{\gamma_m I_n(\delta_k a)K_{n-1}(\gamma_m a) + \delta_k I_{n-1}(\delta_k a)K_n(\gamma_m a)}{\delta_k^2 - \gamma_m^2}$$

$$\sum_{m=1}^{\infty} b_m(\delta_n^2+\gamma_m^2)^{-1} = K(i\delta_n) = 0, \qquad n = 1, 2, \ldots$$

and equating to zero the coefficients of the linearly independent functions $J_n(i\gamma_m r)$, we obtain an infinite algebraic system for determining the coefficients C_k $(k=1, 2\ldots)$ of the expansion (2.7)

$$\sum_{k=1}^{\infty} C_k(\delta_k^2-\gamma_m^2)^{-1}[\gamma_m I_n(\delta_k a)K_{n-1}(\gamma_m a)+\delta_k I_{n-1}(\delta_k a)K_n(\gamma_m a)] = \tag{2.10}$$
$$i^{-n}K^{-1}(\varepsilon)(\varepsilon^2+\gamma_m^2)^{-1}[\gamma_m J_n(\varepsilon a)K_{n-1}(\gamma_m a)+\varepsilon J_{n-1}(\varepsilon a)K_n(\gamma_m a)]$$
$$m=1, 2, \ldots$$

($I_n(x)$ and $K_n(x)$ are, respectively, the Bessel function of an imaginary argument and the MacDonald function).

In [3] the system (2.10) was investigated by reducing it to a system of the second kind through an exact inversion of the principal singular part; this was justified by the method of successive approximations for large δ_k and γ_m $(k, m=1, 2\ldots)$ or for large a.

We restrict ourselves to the principal term in the asymptotic solution of the system (2.10) for $\delta_k, \gamma_m \to \infty$, $(k, m = 1, 2 \ldots)$. For this purpose we introduce the new unknowns

$$C_k = 2X_n K_n(\delta_k a)\delta_k a \tag{2.11}$$

and perform passage to the limit for $\delta_k, \gamma_m \to \infty$. We thus arrive at the infinite system

$$\sum_{k=1}^{\infty}\frac{X_k}{\delta_k-\gamma_m} = i^{-n}K^{-1}(\varepsilon)\frac{\gamma_m J_n(\varepsilon a)+\varepsilon J_{n-1}(\varepsilon a)}{\varepsilon^2+\gamma_m^2}, \quad m=1, 2, \ldots \tag{2.12}$$

We write the solution of this system in the form [3]

$$X_k = \frac{i[J_n(\varepsilon a)+iJ_{n-1}(\varepsilon a)]}{2K_+(\varepsilon)(\delta_k+i\varepsilon)K_+'(-i\delta_k)}+\frac{i[J_n(\varepsilon a)-iJ_{n-1}(\varepsilon a)]}{2K_-(\varepsilon)(\delta_k-i\varepsilon)K_+'(-i\delta_k)} \tag{2.13}$$

$$K(\alpha)=K_+(\alpha)K_-(\alpha)$$

Here $K_+(\alpha)$ and $K_-(\alpha)$ are functions regular in the upper and lower halfplanes, respectively. If the right side of the first relation of the dual equation (2.1) is r^{2k} and if $n=0$ in (2.1), then the solution of this dual equation is the function

$$Q^*(\alpha)=\lim_{\varepsilon\to 0} L_\varepsilon^k Q(\alpha), \tag{2.14}$$

where L_ε^k denotes the k-fold real operator L_ε with respect to the variable ε.

We now transform the first relation in (1.2) by applying to it the operator L_r^{-1}, which is the operator inverse to L_r. Assuming φ to be bounded for $r=0$, we have (setting $a=1$)

$$\int_0^\infty f(\alpha)K(\alpha)J_0(\alpha r)\alpha\, d\alpha = U\left(\frac{r^2}{4}-c_1\right), \quad r\leqslant 1 \tag{2.15}$$
$$\int_0^\infty f(\alpha)J_0(\alpha r)\alpha\, d\alpha = 0, \quad r>1$$

Similarly, in the case of the off-center impact the right side of the first relation has the form

$$-\omega\left({}^1/_{32}\, r^4+{}^1/_4\, cr^2+c_2\right)$$

Here c_1 and c_2 are constants of integration.

Based on the relations (2.8), (2.11), (2.13) and (2.14), we can now construct the principal terms of the asymptotic solution of the transformed dual integral equations for $\delta_k, \gamma_m \to \infty$ $(k, m = 1, 2 \ldots)$, and, since for the problem in question

$$\delta_k = \frac{\pi k}{h}, \qquad \gamma_m = \frac{\pi}{h}\left(m - \frac{1}{2}\right) \tag{2.16}$$

we have, consequently, the result for $h \to 0$.

3. Calculation of the impact pressure, the apparent additional mass and moment of inertia. For $z = 0$ we have

$$\varphi(r, 0) = \lim_{\varepsilon\to 0} U\,[{}^1/_4 L_\varepsilon q(r) - c_1 q(r)] \tag{3.1}$$

$$\Phi(r, 0) = -\lim_{\varepsilon\to 0} \omega\,[{}^1/_{32} L_\varepsilon^2 q(r) + {}^1/_4 c L_\varepsilon q(r) + c_2 q(r)]$$

Here $q(r)$ is the Hankel transform (2.6) of the solution of Eq. (2.1) for $n = 0$, written in the form (2.7). Omitting the cumbersome derivation, we write the following result:

$$q(r)\,|_{\varepsilon=0} = h^{-1} + \sum_{k=1}^{\infty} C_k^\circ I_0(\delta_k r) \tag{3.2}$$

$$L_\varepsilon q(r)\,|_{\varepsilon=0} = \frac{r^2}{h} - \frac{4h}{3} + \sum_{k=1}^{\infty} B_k^\circ I_0(\delta_k r)$$

$$L_\varepsilon^2 q(r)\,|_{\varepsilon=0} = \frac{r^4}{h} - \frac{16r^2 h}{3} - \frac{64h^3}{45} + \sum_{k=1}^{\infty} D_k^\circ I_0(\delta_k r)$$

Here

$$B_k^\circ = C_k^\circ (d_0 + d_1\delta_k^{-1} + 4\delta_k^{-2})$$

$$D_k^\circ = {}^8/_3 C_k^\circ (b_0 + b_1\delta_k^{-1} + b_2\delta_k^{-2} + b_3\delta_k^{-3} + 24\delta_k^{-4})$$

$$C_k^\circ = \lim_{\varepsilon\to 0} C_k = 2i\,[K_+(0)\,K_+'(-i\delta_k)]^{-1} K_0(\delta_k)$$

$$d_0 = 1 + 2\pi^{-1} h \ln 4 - {}^2/_3 h^2 + 2\pi^{-2} h^2 \ln^2 4$$

$$d_1 = 2 + 4\pi^{-1} h \ln 4$$

$$b_0 = {}^3/_8 + {}^3/_2 a_1 h + h^2(6a_1^2 - 3a_2) + h^3(2a_3 - 12a_1a_2 + 12a_1^3) + h^4(-a_4 + 8a_1a_3 + 6a_2^3 - 36a_1^2a_2 + 24a_1^4)$$

$$b_1 = {}^3/_2 + 6ha_1 + h^2(12a_1^2 - 6a_2) + h^3(4a_3 + 24a_1^3 - 24a_1a_2)$$

$$b_2 = 6 + 12ha_1 + h^2(24a_1^2 - 12a_2)$$

$$b_3 = 12 + 24ha_1$$

$$a_1 = \pi^{-1}\ln 4, \qquad a_2 = \pi^{-2}\ln^2 4 + {}^1/_3$$

$$a_3 = \pi^{-3}\ln^3 4 + \pi^{-1}\ln 4 + 12\pi^{-3} a_5$$

$$a_4 = \pi^{-4}\ln^4 4 + 2\pi^{-2}\ln^2 4 + 48\pi^{-4} a_5 + {}^{19}/_{15}$$

$$a_5 = 1.2020569$$

It is easily seen that the series (3.2) diverge for $r = 1$. Therefore, imposing on φ and ψ the condition of boundedness, we make it possible to determine the constants c_1, c_2 and c. We find that

$$c_1 = \frac{1}{4} + \frac{h \ln 4}{2\pi} - \frac{h^2}{6} + \frac{h^2 \ln^2 4}{2\pi^2} - \frac{d_0}{4} \tag{3.3}$$
$$c = -\frac{b_1}{3d_1}, \qquad c_2 = \frac{b_1 d_0}{12 d_1} - \frac{b_0}{12}$$

On the basis of the relations (1.4), (3.1) – (3.3) we obtain the final expressions for the potential of the velocities $(z = 0,\ r \leqslant 1)$ acquired by the liquid particles as the result of the impact

$$\varphi(r, 0)|_{r\leqslant 1} = U\left(\frac{r^2-1}{4h} - \frac{\ln 4}{2\pi} - \frac{h\ln^2 4}{2\pi^2} - \frac{h}{6} + \frac{1}{\pi}\sum_{k=1}^{\infty} S_k I_0(\delta_k r)\right) \tag{3.4}$$

$$\psi(r, 0)|_{r\leqslant 1} = -\omega x\left(\frac{r^2}{8h} - \frac{b_1}{6d_1 h} - \frac{h}{3} + \frac{1}{6hr}\sum_{k=1}^{\infty} R_k I_1(\delta_k r)\right) \tag{3.5}$$

$$S_k = \frac{[k(\pi + 2h\ln 4) + 2h]\, K_0(\delta_k)}{k(2k)!!\,[(2k-1)!!]^{-1}},$$
$$R_k = \frac{(2k-1)!!}{(2k)!!}\left(b_2 - \frac{4b_1}{d_1} + \frac{hb_3}{\pi k} + \frac{b_4 h^2}{\pi^2 k^2}\right) K_0(\delta_k)$$

Here we have used the fact that

$$K_+(\alpha) = \sqrt{\frac{h}{\pi}}\,\Gamma\left(\frac{1}{2} - \frac{ih\alpha}{\pi}\right)\Gamma^{-1}\left(1 - \frac{ih\alpha}{\pi}\right), \qquad K_-(\alpha) = K_+(-\alpha)$$
$$\frac{i}{K_+'(-i\delta_k)} = \frac{\pi}{h\sqrt{h}}\,\frac{(2k-1)!!}{2^k(k-1)!}$$

The impact pressure $p_l = -\rho(\varphi + \psi)$, where ρ is the density of the liquid. Using the relations (3.4) and (3.5), we obtain expressions for calculating the impact pressure on the boundary of the liquid beneath the disk for small values of h.

The total impact momentum P and the total moment M of the impact pressures acting on the disk are given by the relations

$$P = -2\pi\rho\int_0^1 r\varphi(r, 0)\,dr = \rho U\left(\frac{\pi}{8\pi} + \frac{\ln 4}{2} + \frac{h\ln^2 4}{2\pi} + \right. \tag{3.6}$$
$$\left. \frac{\pi h}{6} - \frac{2}{\pi}\sum_{k=1}^{\infty} k^{-1} S_k I_1(\delta_k)\right)$$

$$M = -\rho\int_0^{2\pi}\int_0^1 x\psi(r, 0)\, r\, d\theta\, dr = \pi\rho\omega\left(\frac{1}{48h} - \frac{b_1}{24 d_1 h} - \right. \tag{3.7}$$
$$\left. \frac{h}{12} + \frac{6}{\pi}\sum_{k=1}^{\infty} k^{-1} R_k I_2(\delta_k)\right)$$

The expressions (3.4) – (3.7) are valid for small values of h.

In Table 1 we present the dimensionless values of the impact pressure at the center of the disk and also the apparent additional mass and moment of inertia

$$p^* = -U^{-1}\varphi(0, 0), \quad P^* = (\rho U)^{-1} P, \qquad M^* = -(\rho\omega)^{-1} M \tag{3.8}$$

for various values of h, calculated, respectively, from the equations (3.4), (3.6) and (3.7). For comparison we present in the columns 3, 5 and 7 of the table analogous results calculated on the basis of the results obtained in [2], which are valid for $h \geqslant 1.1$ with a relative error to 6%.

Table 1

h	p^*		P^*		M^*	
1	2	3	4	5	6	7
1.5			1.408	1.365	0.2263	0.1780
1.1	0.4985	0.6750	1.448	1.420	0.2004	0.1780
1.0	0.5303	0.6890	1.465	1.461	0.1961	0.1774
0.9	0.5641	0.7133	1.488	1.541	0.1926	0.1756
0.8	0.6019		1.518		0.1902	
0.7	0.7559		1.561		0.1892	
0.6	0.7930		1.624		0.1898	
0.5	0.8518		1.720		0.1933	

4. Condition for a nonseparable impact. The coordinate of the point of application of the momentum is given by

$$x_0 = M / P \tag{4.1}$$

Separation of the disk from the liquid surface occurs if, at least in the neighborhood of the point $x = 1$, $y = 0$, the impact pressure becomes negative (we assume that $\omega > 0$, $U > 0$), i.e. if

$$(\varphi + \psi) \geqslant 0 \quad \text{for} \quad z = 0, \quad x \to 1, \quad y = 0 \tag{4.2}$$

Taking relation (4.1) into account, we obtain from (4.2) the condition for nonseparation of the disk from the liquid surface at impact

$$|x_0| \leqslant -\lim_{r \to 1} \frac{M\varphi}{P\psi} = \beta \quad \text{for} \quad z = 0, \quad y = 0, \quad 0 < x \leqslant 1 \tag{4.3}$$

Using the behavior of the functions $K_0(x)$, $I_0(x)$, $I_1(x)$ and $I_2(x)$ for large values of the argument, we find asymptotic values of the expressions (3.4) and (3.5) in the neighborhood of the point $r = 1$, and of the expressions (3.6) and (3.7) for $h \to 0$. We have

$$\varphi(r, 0)|_{r \leqslant 1} = U\left[\sqrt{\frac{1-r}{\pi h}} + O(1)\right] \tag{4.4}$$

$$\psi(r, 0)|_{r \leqslant 1} = \omega x\left[\frac{1}{2}\sqrt{\frac{1-r}{\pi h}} + O(1)\right]$$

$$P = \rho U\left[\frac{\pi}{8h} + O(1)\right]$$

$$M = -\rho\omega\left[\frac{\pi}{96h} + O(1)\right]$$

From the condition (4.3) on the basis of (4.4) we have $\beta = {}^1/_6$ for $h \to 0$. Finally, the condition for a nonseparable impact for $h \to 0$ becomes

$$|x_0| \leqslant a / 6$$

In the case $h = \infty$ the condition for a nonseparable impact is given by $|x_0| \leqslant a / 5$ (see [2]). Consequently, we can write the condition for a nonseparable impact onto an ideal liquid for arbitrary values of h in the form

$$|x_0| \leqslant a^*(h), \ a / 6 \leqslant a^*(h) \leqslant a / 5, \ 0 < h < \infty$$

$$a^*(h) \to a/5 \quad \text{for} \quad h \to \infty, \qquad a^*(h) \to a/6 \quad \text{for} \quad h \to 0$$

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BOUNDARY LAYER IN THE PROBLEM OF LONGITUDINAL MOTION OF A CYLINDER IN A VISCOPLASTIC MEDIUM

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P. P. MOSOLOV and V. P. MIASNIKOV
(Moscow)
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Construction of the boundary layer by the method of variation is used for investigating the principle of selecting the unique solution for a perfectly plastic medium by transition to it from a viscoplastic medium with the viscosity coefficient tending to zero.

Let an infinitely long cylinder move along its axis in a viscoplastic medium at constant velocity. The velocity field of particles of a viscoplastic medium induced by the cylinder motion in a system of coordinates x, y, z (with the cylinder axis along the z-coordinate and its cross section ω lying in xy-plane) is of the form $u = (0, 0, u(x, y))$. It was shown in [1] that $u(x, y)$ minimizes functional

$$I_1(w) = \int_{R^2\setminus\omega} \left[\frac{\mu}{2}|\nabla w|^2 + \tau_0|\nabla w|\right] dw - Fw|_{d\omega}, \quad w|_{d\omega} = \text{const}$$

where μ and τ_0 are, respectively, the viscosity coefficient and the yield point of the medium and F is the longitudinal force moving the cylinder. The velocity of the cylinder can be determined when force F is specified. It is $u(x, y)$ over $\partial\omega$. If the cylinder velocity is u_0, then $u(x, y)$ minimizes functional

$$I_2(w) = \int_{R^2\setminus\omega} \left[\frac{\mu}{2}|\nabla w|^2 + \tau_0|\nabla w|\right] dw, \qquad w|_{\partial\omega} = u_0 \tag{1}$$

and the force necessary for producing such motion is determined by formula

$$u_0 F = I_2(u) + \int_{R^2/\omega} \frac{\mu}{2}|\nabla u|^2 d\omega$$